On Type Systems For Probabilistic Termination

Ugo Dal Lago
(Based on joint work with
Charles Grellois and Flavien Breuvart)

CRECOGI+ELICA+GDRILL Meeting, October 10th 2018

Deterministc vs. Probabilistic Transition Systems

$$M$$

Deterministc vs. Probabilistic Transition Systems

$$M \longrightarrow N$$

Deterministc vs. Probabilistic Transition Systems

$$M \longrightarrow N \longrightarrow L$$

Deterministc vs. Probabilistic Transition Systems

$$M \longrightarrow N \longrightarrow L \longrightarrow \cdots$$

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$$(\mathcal{A}, \longrightarrow\!\!\blacktriangleright)$$

$$\longrightarrow\!\!\blacktriangleright : \mathcal{A} \rightharpoonup \mathcal{A}$$

Deterministc vs. Probabilistic Transition Systems

M

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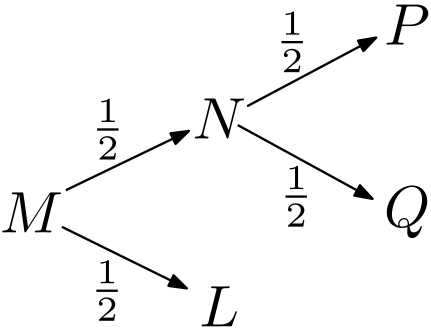
$$\begin{tikzcd} & N \\ M \arrow[ur, "\frac{1}{2}"] \arrow[dr, "\frac{1}{2}"'] & \\ & L \end{tikzcd}$$

Deterministc vs. Probabilistic Transition Systems



Deterministc vs. Probabilistic Transition Systems

$$
\begin{tikzcd}
& & P \\
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\end{tikzcd}
$$

$$(\mathcal{A}, \longrightarrow)$$

$$\longrightarrow : \mathcal{A} \rightharpoonup \mathsf{Dist}(\mathcal{A})$$

The Landscape: *Type* Theory

Simple Types

$$\tau ::= \iota \quad \Big| \quad \tau \to \tau$$

The Landscape: *Type* Theory

Simple Types

$$\tau ::= \iota \quad | \quad \tau \to \tau$$

- Sound for termination, in absence of recursion.
- Poor expressive power.
- Intuitionistic Logic.

The Landscape: *Type* Theory

Simple Types

$$\tau ::= \iota \;\Big|\; \tau \to \tau$$

Polymorphic Types

$$\tau ::= \cdots \;\Big|\; \alpha \;\Big|\; \forall \alpha . \tau$$

- Second-order Intuistionistic Logic.
- Very expressive, extensionally.
- Still poor, intensionally.

ypes

$\tau \to \tau$

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Intersection Types

$$\tau ::= \cdots \mid \tau \wedge \tau$$

The Landscape: *Type* Theory

- Motivated by Semantics.
- *Complete* for termination.
- Type inference is undecidable.

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Sized Types

$$\tau ::= \cdots \;\big|\; \iota[\xi]$$

The Landscape: *Type* Theory

- Reasonably expressive, intensionally.
- Type inference remains decidable

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$\tau ::= \cdots \mid \iota[\xi]$

The Landscape: *Recursion* Theory

Determinism

$M\overline{s} \to^* N_s$

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Determinism	**Probabilism**
$M\overline{s} \to^* N_s$	$[\![M\overline{s}]\!] = \mathcal{D}_s$

The Landscape: *Recursion* Theory

$\sum \mathcal{D}_s$ can be smaller than 1.

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Undecidable;
Σ^0_1-complete.

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Almost-Sure Termination
Π^0_2-complete.

	De	
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- Pure λ-calculus with simple types is terminating.
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- For every type τ, define a set of reducible terms Red_τ.
- Prove that all reducible terms are normalizing…
- …and that all typable terms are reducible.

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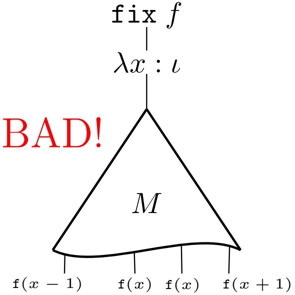

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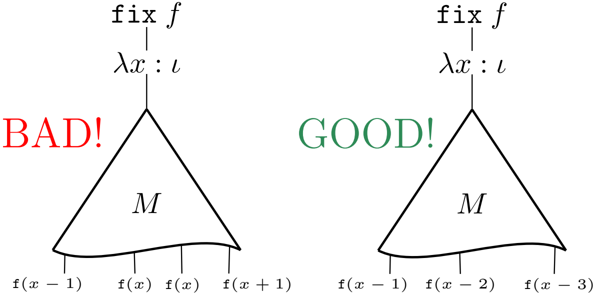

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$$\xi ::= a \;\Big|\; \omega \;\Big|\; \xi + 1; \qquad\qquad \tau ::= \iota[\xi] \;\Big|\; \tau \to \tau.$$

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Index Terms

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  - ... but of an indexed form.

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- **Typing Fixpoints**.

- Reducibility sets are of the form Red_{τ}^{θ}.
- θ is an environment for index variables.
- Proof of reducibility for $\texttt{fix}\, x.M$ is rather delicate.

- **Q**
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- **Termination**.
  - Proved by **Reducibility**.
  - ... but of an indexed form.
- **Type Inference**.
  - It is indeed *decidable*.
  - But *nontrivial*.

Probabilistic Termination

- **Examples**:

  $\texttt{fix}\ f.\lambda x.\texttt{if}\ x > 0\ \texttt{then}\ \texttt{if}\ \mathit{FairCoin}\ \texttt{then}\ f(x-1)\ \texttt{else}\ f(x+1);$

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  Unbiased Random Walk

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Biased Random Walk, the “wrong” way.

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- Probabilistic termination **is** thus:
  - Sensitive to *the actual distribution* from which we sample.
  - Sensitive to *how many recursive calls* we perform.

One-Counter Blind Markov Chains

- They are automata of the form (Q, δ) where
  - Q is a finite set of *states*.
  - $\delta : Q \to \mathsf{Dist}(Q \times \{-1, 0, 1\})$.
- They are a very special form of One-Counter Markov Decision Processeses [BBEK2011].
  - Everything is purely deterministic.
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  - Everything is purely deterministic.
  - The counter value is ignored.
- The probability of reaching a configuration where the counter is 0 can be approximated arbitrarily well *in polynomial time*.

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Every higher-order variable occurs **at most once**.

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This is sufficient for typing:

- Unbiased random walks;
- Biased random walks.

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- **Typing Probabilistic Choice**

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  - Reducibility sets are now on the form $Red_{\tau}^{\theta,p}$
  - p stands for the *probability* of being reducible.
  - Reducibility sets are continuous:

$$Red_{\tau}^{\theta,p} = \bigcup_{q<p} Red_{\tau}^{\theta,q}$$

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Part I

Intersection Types

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- **Termination**
  - Again by reducibility.
- **Completeness**
  - By *subject expansion*, the dual of subject reduction.

Oracle Intersection Types [BreuvartDL2018]

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This is **unavoidable**, due to recursion theory.

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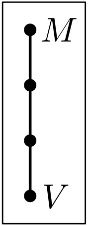
Intersection Types and Computations



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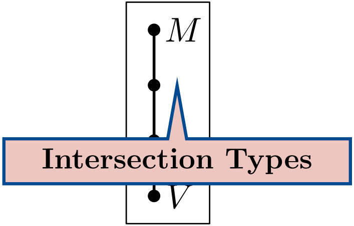

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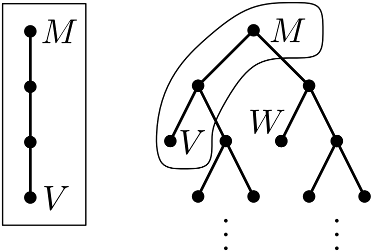

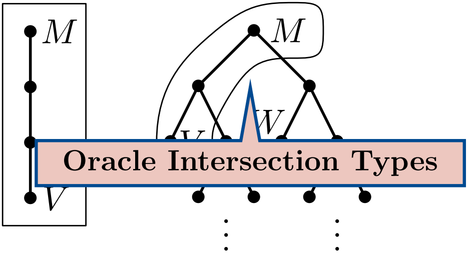
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Oracle Intersection Types

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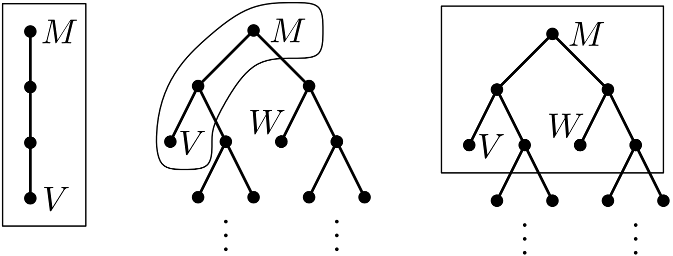
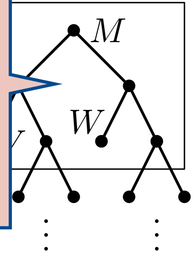


Intersection Types and Computations

Monadic Intersection Types [BDL2018]

- They are a combination of oracle and sized types.
- Intersections are needed for preciseness.
- Distributions of types allow to analyse more than one probabilistic branch in the same type derivation.



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  - In linear dependent types [DLG2011], one is (relatively complete) for deterministic first-order functions.
  - How about probabilism?
    - Monadic types becomes indexed:

$$\mu ::= \{\sigma[i] : p[i]\}_{i \in I}$$

    - Subtyping is coupling-based.

Thank You!

Questions?