Co-lifting of Heyting algebras and its S4 analogue

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Introduction

In logic:

$$\text{Intuitionistic logic} \xrightarrow{\text{Girard translation}} \text{Modal logic S4}$$

In algebra:

$$\text{Heyting algebras} \xleftarrow{\text{Kleisli construction}} \text{S4 algebras}$$

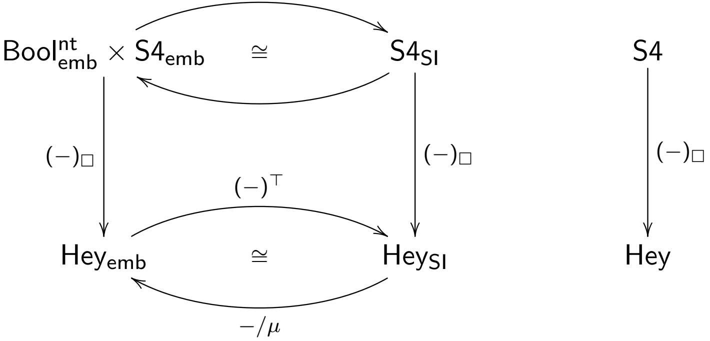
This can be extended to all intermediate logics and extensions of the modal logic S4.
(A logic L is called an intermediate logic if $Int \subseteq L \subseteq Cl$)

Introduction

- A. Wroński, "Intermediate logics and Disjunction Property" (1973)
  'There are 2^ω intermediate logics with the disjunction property'.
  In the paper, two constructions of Heyting algebras appear:
  - co-lifting of Heyting algebras $\mathbb{H}^\top$
  - quotient modulo the monolith $\mathbb{H}/\mu_\mathbb{H}$
- We propose an S4 analogue of these constructions.
- We show that they correspond to each other via the Kleisli construction.
- As an application, we get new logics having the disjunction property.

Definition

S4 : the category of S4 algebras and homomorphisms.
Hey : the category of Heyting algebras and homomorphisms.



Outline

Outline

Heyting algebra

Definition

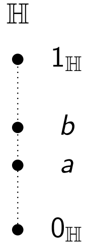
A Heyting algebra $\mathbb{H} = \langle H, \wedge, \vee, 1, 0, \to \rangle$ is a bounded distributive lattice equipped with a binary operation $\to$ s.t. $a \wedge b \leq c \Leftrightarrow a \leq b \to c$.

Remark: any Heyting algebra can be seen as a posetal Cartesian closed category with finite coproducts.

Example:

Any bounded chain can be regarded as a Heyting algebra with

$$a \to b = \begin{cases} 1 & a \leq b \\ b & o.w. \end{cases}$$



S4 algebra

Definition

An S4 algebra $\mathbb{B} = \langle B, \wedge, \vee, -, 1, 0, \Box\rangle$ is a Boolean algebra equipped with a unary operation $\Box$ s.t. $\Box(a \wedge b) = \Box a \wedge \Box b$, $\Box 1 = 1$, $\Box a \leq a$ and $\Box a \leq \Box\Box a$.

Remark: any S4 algebra can be seen as a posetal category with a comonad $\Box$ which preserves finite products.

Example:
Let $\mathbb{X} = \langle X, \tau\rangle$ be a topological space and int an interior operator.
Then $\langle \mathcal{P}(X), \cap, \cup, \sim, X, \emptyset, \mathrm{int}\rangle$ is an S4 algebra.

Kleisli construction

Definition

$\mathbb{B}$: an S4 algebra.

- A Kleisli order $\leq_\Box$ on $\mathbb{B}$ is defined by $a \leq_\Box b \Leftrightarrow \Box a \leq b$.
- $B_\Box := \langle B, \leq_\Box \rangle / \sim$ where $a \sim b \Leftrightarrow a \leq_\Box b$ and $b \leq_\Box a$.
- $\mathbb{B}_\Box := \langle B_\Box, \vee_\Box, \wedge_\Box, 1_\Box, 0_\Box, \to_\Box \rangle$ where
  - $[a] \wedge_\Box [b] = [a \wedge b]$,
  - $[a] \vee_\Box [b] = [\Box a \vee \Box b]$,
  - $1_\Box = [1]$,
  - $0_\Box = [0]$, and
  - $[a] \to_\Box [b] = [\Box a \to b]$.

Kleisli construction

Lemma

$\mathbb{B}$: an S4 algebra.
Then, $\mathbb{B}_\Box$ is a Heyting algebra.

Remark: $\mathbb{B}_\Box$ is a Kleisli category of $\Box$. (In this case, the Kleisli-cat. and the EM-cat. of a posetal category $\mathbb{B}$ are categorically equivalent.)

The Kleisli construction can be extended to a functor $(-)_\Box : \mathbf{S4} \longrightarrow \mathbf{Hey}$.

Lemma

$\mathbb{B}_1, \mathbb{B}_2$: S4 algebras, $f : \mathbb{B}_1 \to \mathbb{B}_2$: an S4 homomorphism.
Define a map $f_\Box : B_{1\Box} \to B_{2\Box}$ by $f_\Box([a]_1) = [f(a)]_2$.
Then $f_\Box$ is a Heyting homomorphism.

Kleisli construction

Definition

S4 : the category of S4 algebras and homomorphisms.
Hey : the category of Heyting algebras and homomorphisms.

$\mathrm{S4}_{\mathrm{emb}}$: the category of S4 algebras and embeddings.
$\mathrm{Hey}_{\mathrm{emb}}$: the category of Heyting algebras and embeddings.

Kleisli construction

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$\mathrm{S4}_{\mathrm{emb}}$: the category of S4 algebras and embeddings.
$\mathrm{Hey}_{\mathrm{emb}}$: the category of Heyting algebras and embeddings.

$$\mathrm{Bool}^{\mathrm{nt}}_{\mathrm{emb}} \times \mathrm{S4}_{\mathrm{emb}} \xrightarrow{(-)_{\Box}} \mathrm{Hey}_{\mathrm{emb}}$$

$$\mathrm{S4}_{\mathrm{SI}} \qquad \mathrm{Hey}_{\mathrm{SI}}$$

$$\mathrm{S4} \xrightarrow{(-)_{\Box}} \mathrm{Hey}$$

Kleisli construction

The Kleisli construction $(-)^{\Box} : \mathsf{S4} \longrightarrow \mathsf{Hey}$ preserves

- congruence lattices and
- subdirect irreducibility.

Congruence lattice

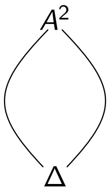
Definition (Universal algebra)

$\mathbb{A}$: an algebra.

- An equivalence relation θ on A is called congruence on $\mathbb{A}$ if it preserves algebraic operations of $\mathbb{A}$.
- Write **Con**$(\mathbb{A})$ for the congruence lattice of $\mathbb{A}$.

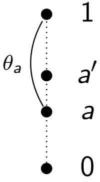
In general:

Con$(\mathbb{A})$

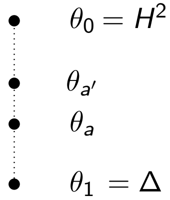


Example: bounded chain (as a Heyting algebra).

$\mathbb{H}$



Con$(\mathbb{H})$



Congruence lattice

The Kleisli construction $(-)_\Box$: S4 $\longrightarrow$ Hey preserves congruence lattices.

Lemma

$\mathbb{B}$ *: an S4 algebra,* $\theta \in \mathbf{Con}(\mathbb{B})$.

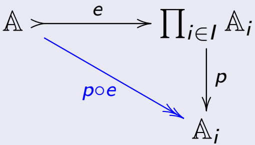
- $\mathbf{Con}(\mathbb{B}) \cong \mathbf{Con}(\mathbb{B}_\Box)$
- $(\mathbb{B}/\theta)_\Box \cong \mathbb{B}_\Box/\theta_\Box$

Subdirectly irreducible

Definition (Universal algebra)

$\mathbb{A}$, $\mathbb{A}_i$ $(i \in I)$: algebras.

- We call $\mathbb{A}$ a subdirect product of $\{\mathbb{A}_i\}_{i \in I}$ if $e : \mathbb{A} \rightarrowtail \prod_{i \in I} \mathbb{A}_i$ where each $p_i \circ e$ is surjective.



- $\mathbb{A}$ is called subdirectly irreducible (s.i. for short) if $\mathbb{A}$ cannot be decomposed as a subdirect product.

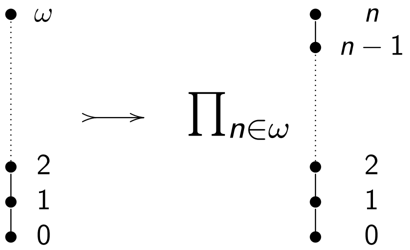
Example: Every finite chain is s.i. as a Heyting algebra.

Subdirectly irreducible

Theorem (Fundamental Theorem of Universal Algebra)

Every algebra is isomorphic to a subdirect product of s.i. algebras.

Example:



Monolith

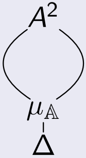
Definition (Universal algebra)

$\mathbb{A}$: an algebra.
A monolith of $\mathbb{A}$ (write $\mu_{\mathbb{A}}$) is the 2nd-minimum congruence of $\mathbb{A}$ (if any).

Lemma (Universal algebra)

$\mathbb{A}$: an algebra.
$\mathbb{A}$ is s.i. $\Leftrightarrow$ $\mathbb{A}$ has a monolith.

Con$(\mathbb{A})$



The Kleisli construction $(-)_\square$: S4 $\longrightarrow$ Hey preserves s.i.

Corollary

$\mathbb{B}$: an S4 algebra. Then, $\mathbb{B}$ is s.i. $\Leftrightarrow$ $\mathbb{B}_\square$ is s.i.

Opremum

Definition

$\mathbb{H}$: a Heyting algebra.
An opremum of $\mathbb{H}$ (write $\star_{\mathbb{H}}$) is the 2nd-largest element of $\mathbb{H}$ (if exists).

Definition

$\mathbb{B}$: an S4 algebra.

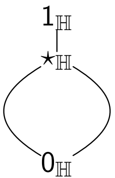
- An opremum of $\mathbb{B}$ is (intuitively) a 2nd-largest element of $\mathbb{B}$ according to the Kleisli order $\leq_{\Box}$
  (i.e. an elements $a \in \mathbb{B} \setminus \{1_B\}$ s.t. $\forall b \in B \setminus \{1_B\}\ \Box b \leq a$ if exists).
- Write $Op(\mathbb{B})$ for the set of oprema of $\mathbb{B}$.
- Write $\star_{\mathbb{B}}$ for the minimum opremum of $\mathbb{B}$ (if exists).
  ($Op(\mathbb{B}) \neq \emptyset \Rightarrow \star_{\mathbb{B}}$ exists.)

Opremum

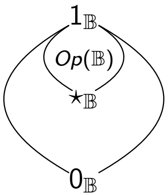
Lemma

$\mathbb{A}$: an algebra (Heyting or S4).
$\mathbb{A}$ is s.i. $\Leftrightarrow$ $\mathbb{A}$ has a monolith $\Leftrightarrow$ $\mathbb{A}$ has an opremum.

s.i. Heyting algebra



s.i. S4 algebra



The Kleisli construction $(-)_{\square}$: S4 $\longrightarrow$ Hey preserves oprema.

Lemma

$\mathbb{B}$: an s.i. S4 algebra. Then, $\star_{\mathbb{B}_{\square}} = Op(\mathbb{B}) = [\star_{\mathbb{B}}]$.

What we have obtained so far

Definition

$\mathbf{S4}_{\mathrm{SI}}$: the category of s.i. S4 algebras and opremum preserving homomorphisms.
$\mathbf{Hey}_{\mathrm{SI}}$: the category of s.i. Heyting algebras and opremum preserving homomorphisms.

$$\mathbf{S4}_{\mathrm{emb}} \xrightarrow{(-)_\Box} \mathbf{Hey}_{\mathrm{emb}} \qquad \mathbf{S4}_{\mathrm{SI}} \xrightarrow{(-)_\Box} \mathbf{Hey}_{\mathrm{SI}} \qquad \mathbf{S4} \xrightarrow{(-)_\Box} \mathbf{Hey}$$

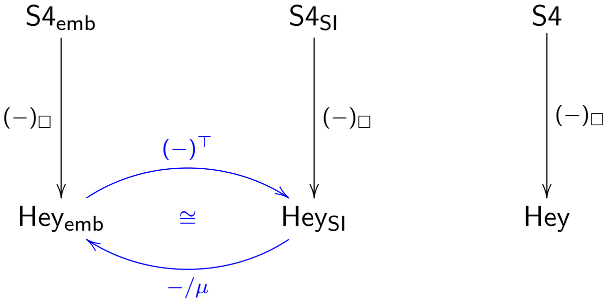
Outline

Co-lifting of Heyting algebras

Definition

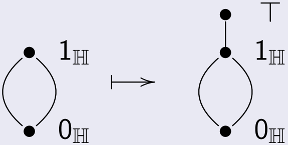
$\mathsf{Hey}_{\mathsf{emb}}$: the category of Heyting algebras and embeddings.
$\mathsf{Hey}_{\mathsf{SI}}$: the category of s.i. Heyting algebras and opremum-preserving homomorphisms.

Here, we will describe the following equivalence:



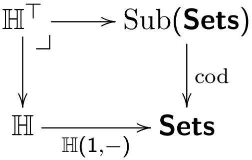
Definition

$\mathbb{H}$: a Heyting algebra.
A co-lifting of $\mathbb{H}$ (write $\mathbb{H}^{\top}$) is obtained by adding a new top-element above the top of $\mathbb{H}$.



This is an s.i. Heyting algebra.

Remark: $\mathbb{H}^{\top}$ is the subscone (injective scone) of $\mathbb{H}$:

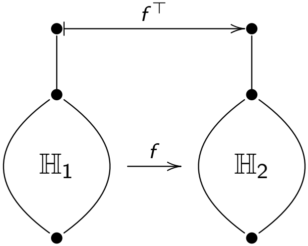


Co-lifting $(-)^{\top}$: $\mathsf{Hey}_{\mathsf{emb}} \longrightarrow \mathsf{Hey}_{\mathsf{SI}}$

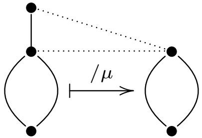
Co-lifting can be extended to a functor $(-)^{\top} : \mathsf{Hey}_{\mathsf{emb}} \longrightarrow \mathsf{Hey}_{\mathsf{SI}}$.

Lemma

$\mathbb{H}_1, \mathbb{H}_2$: Heyting algebras, $f : \mathbb{H}_1 \to \mathbb{H}_2$: Heyting homomorphism. If f is an embedding, then the natural extension $f^{\top} : \mathbb{H}_1^{\top} \to \mathbb{H}_2^{\top}$ is an opremum-preserving hom.



Quotient modulo the monolith:



Lemma

$\mathbb{H}$: *an s.i. Heyting algebra.*
$(\mathbb{H}/\mu_{\mathbb{H}})^{\top} \cong \mathbb{H}$.

This can be extended to a functor $-/\mu : \mathsf{Hey_{SI}} \longrightarrow \mathsf{Hey_{emb}}$.

Lemma

$\mathbb{H}_1, \mathbb{H}_2$: *s.i. Heyting algebras,* $f : \mathbb{H}_1 \to \mathbb{H}_2$ *Heyting homomorphism,*
If f *preserves the opremum, then there is* $f' : \mathbb{H}_1/\mu_{\mathbb{H}_1} \rightarrowtail \mathbb{H}_2/\mu_{\mathbb{H}_2}$.

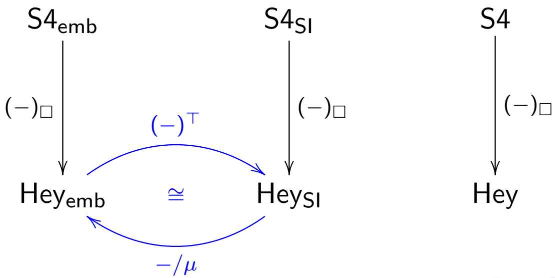
Consequence

Definition

$\mathsf{Hey}_{\mathsf{emb}}$: the category of Heyting algebras and embeddings.
$\mathsf{Hey}_{\mathsf{SI}}$: the category of s.i. Heyting algebras and opremum-preserving homomorphisms.

Theorem

$\mathsf{Hey}_{\mathsf{emb}} \cong \mathsf{Hey}_{\mathsf{SI}}$.



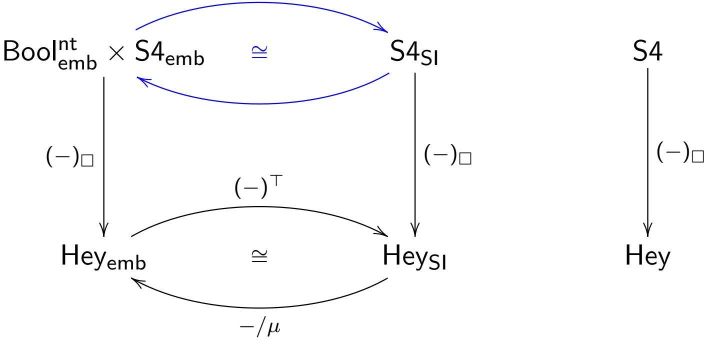
Outline

S4 analogue of co-lifting

Definition

$\mathsf{Bool}^{\mathsf{nt}}_{\mathsf{emb}}$: the category of non-trivial Boolean algebras and embeddings.

Here, we will show the following equivalence and commutativity (up-to-iso) of the diagram:

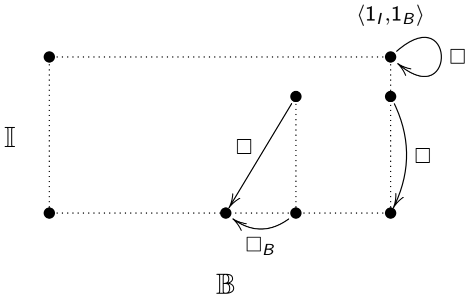


• construction: S4-analogue of co-lifting $(-)^\top$

Definition

$\mathbb{I}$: a non-trivial Boolean algebra, $\mathbb{B} = \langle \mathbb{B}, \Box_B \rangle$: an S4 algebra.
Let $\mathbb{I} \bullet \mathbb{B} := \langle \mathbb{I} \times \mathbb{B}, \Box \rangle$ where $\Box : I \times B \to I \times B$ is defined by

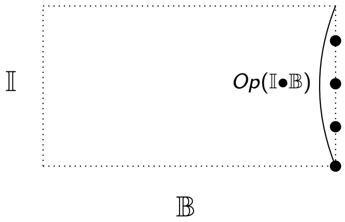
$$\Box\langle i, a\rangle = \begin{cases} \langle 1_{\mathbb{I}}, 1_{\mathbb{B}} \rangle & \langle i, a \rangle = \langle 1_{\mathbb{I}}, 1_{\mathbb{B}} \rangle \\ \langle 0_{\mathbb{I}}, \Box_B a \rangle & o.w. \end{cases}$$



• construction: S4-analogue of co-lifting $(-)^{\top}$

Lemma

$\mathbb{I}$: a non-trivial Boolean algebra, $\mathbb{B}$: an S4 algebra.
$\mathbb{I} \bullet \mathbb{B}$ is an s.i. S4 algebra with $Op(\mathbb{I} \bullet \mathbb{B}) = \{\langle i, 1_{\mathbb{B}} \rangle \mid i \in \mathbb{I} \setminus \{1_{\mathbb{I}}\}\}$



Lemma

$\mathbb{I}$: a non-trivial Boolean algebra, $\mathbb{B}$: an S4 algebra.
$(\mathbb{I} \bullet \mathbb{B})/\mu_{\mathbb{I} \bullet \mathbb{B}} \cong \mathbb{B}$

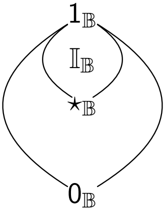
Decomposition into • algebra

Lemma

$\mathbb{B}$: an s.i. S4 algebra.

$\mathbb{I}_{\mathbb{B}} \bullet (\mathbb{B}/\mu_{\mathbb{B}}) \cong \mathbb{B}$ where the Boolean algebra $\mathbb{I}_{\mathbb{B}}$ is defined by

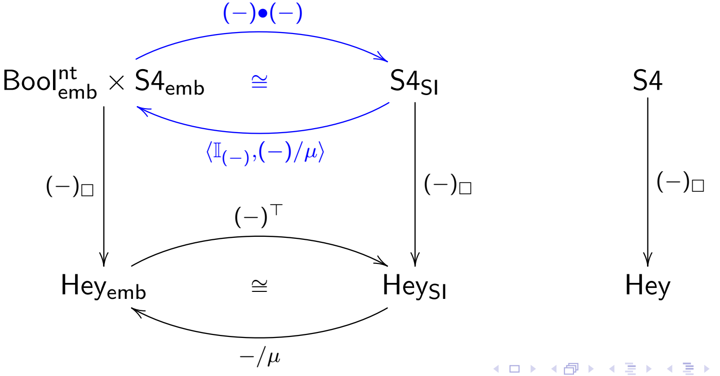
- *base set $I_{\mathbb{B}} := Op(\mathbb{B}) \cup \{1_{\mathbb{B}}\}$*
- *$\wedge, \vee, 1_{\mathbb{I}}$ are same as those of $\mathbb{B}$*
- *$-i = -_{\mathbb{B}} i \vee \star_{\mathbb{B}}$*
- *$0_{\mathbb{I}} = \star_{\mathbb{B}}$*



Consequence

Definition

$\mathsf{Bool}^{\mathsf{nt}}_{\mathsf{emb}}$: the category of non-trivial Boolean algebras and embeddings.
$\mathsf{S4}_{\mathsf{emb}}$: the category of S4 algebras and embeddings.
$\mathsf{S4}_{\mathsf{SI}}$: the category of s.i. S4 algebras and opremum-preserving homomorphisms.

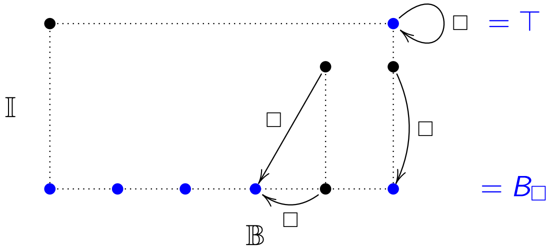


Commutativity of co-lifting and Kleisli construction

Lemma

$\mathbb{I}$: a non-trivial Boolean algebra, $\mathbb{B}$: an S4 algebra.
$(\mathbb{I} \bullet \mathbb{B})_{\Box} \cong (\mathbb{B}_{\Box})^{\top}$.

Remark: $\mathbb{B}_{\Box}$ is same as a sublattice of $\mathbb{B}$ that consists of box-stable elements $\{a \in \mathbb{B} \mid \Box a = a\}$ (i.e. EM-cat. of $\langle \mathbb{B}, \Box \rangle$).



Summary

- We refined the co-lifting of Heyting algebras (appeared in Wrónski 1973) as a categorical equivalence,
- proposed the • construction as S4 analogue of co-lifting,
- and showed correspondence of co-lifting and • construction via the Kleisli construction.

$$\begin{array}{ccccc}
\mathsf{Bool}^{\mathsf{nt}}_{\mathsf{emb}} \times \mathsf{S4}_{\mathsf{emb}} & \cong & \mathsf{S4}_{\mathsf{SI}} & & \mathsf{S4} \\
\downarrow (-)_{\Box} & & \downarrow (-)_{\Box} & & \downarrow (-)_{\Box} \\
\mathsf{Hey}_{\mathsf{emb}} & \cong & \mathsf{Hey}_{\mathsf{SI}} & & \mathsf{Hey}
\end{array}$$

Application

Theorem (Wrónski, 1973)

There are 2^ω intermediate logics having the disjunction property (DP).

There is a map $\tau : \mathbf{Ext}(\mathrm{Int}) \to \mathbf{NExt}(\mathrm{S4})$ preserving the disjunction property. So, as an immediate consequence of Wrónski's theorem, we get:

Corollary

There are 2^ω extensions of the modal logic S4 with the disjunction property.

As an application of our work, we get new logics with DP:

Proposition (new result)

There are infinitely many extensions of S4 which have the disjunction property and cannot be expressed as τL.