EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Proof equivalence in second order multiplicative linear logic

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Outline

1. Proof nets and proof equivalence
2. Proof nets, coends and the Yoneda isomorphism
3. Weak coherence for coends
4. Observational equivalence (joint work with L. Tortora de Falco)

Outline

Proof nets and "bureaucracy"

Proof nets were designed to provide canonical representations of proofs:

- invariant with respect to reductions and permutations

$$\dfrac{\dfrac{A \qquad B, C}{A \otimes B, C} \qquad D}{A \otimes B, C \otimes D} \quad \to_\gamma \quad \dfrac{A \qquad \dfrac{B, C \qquad D}{B, C \otimes D}}{A \otimes B, C \otimes D}$$

- free categories, coherence, etc.

$$\begin{tikzcd} A \otimes (B \parr C) \arrow[r, "\pi_{A,B,C}"] \arrow[d, "A \otimes (B \parr f)"] & (A \otimes B) \parr C \arrow[d, "(A \otimes B) \parr f"] \\ A \otimes (B \parr (C \otimes D)) \arrow[r, "\pi_{A,B,C\otimes D}"'] & (A \otimes B) \parr (C \otimes D) \end{tikzcd}$$

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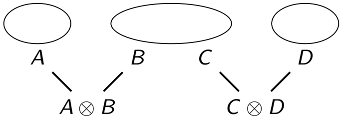

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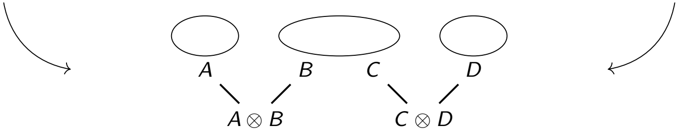

Then for two derivations d, d' it is equivalent:

- d, d' induce the same proof net
- d can be obtained from d' by permuting rules
- d, d' have the same interpretation in all *-autonomous category

Weak coherence: MLL^+ proof nets

The equivalence $\simeq_{Perm}$ for MLL^+ (and the equivalence $\simeq_{Diag}$ given by the free *-autonomous category) is *PSPACE*-complete [HH2014].

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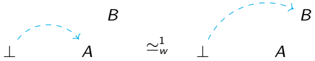
- Define **1-rewiring** $\simeq^1_w$ over proof nets: 

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Example:

$$\perp \parr \perp \quad \mathbf{1} \otimes \mathbf{1} \;\simeq_w\; \perp \parr \perp \quad \mathbf{1} \otimes \mathbf{1} \qquad \mathbf{1} \parr \mathbf{1} \quad \perp \otimes \perp \;\not\simeq_w\; \mathbf{1} \parr \mathbf{1} \quad \perp \otimes \perp$$

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Then d, d' can be obtained by permuting rules iff they induce the same proof net up to rewiring.

Theorem: [BCST1996, Hughes2012] The category of MLL^+ proof nets modulo rewiring is the free *-autonomous category.

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Some results which hold for MLL and $\lambda_{\rightarrow}$ fails for F:

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Moreover, many "wanted" isomorphisms fail for $\beta\eta$:

- Russell-Prawitz translation: $A \vee B \simeq \forall X((A \Rightarrow X) \Rightarrow (B \Rightarrow X) \Rightarrow X)$
- initial algebras, final coalgebras: $\mu XA \simeq \forall X((A \Rightarrow X) \Rightarrow X)$, $\nu XA \simeq \exists X((X \Rightarrow A) \wedge X)$
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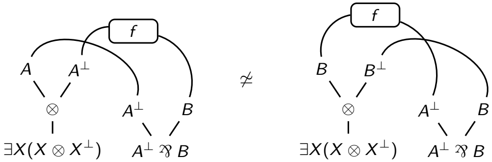
Hence $\beta\eta$-equality is in general too weak and one has to look at **models**:

- **Parametric models** (extensional), characterize wanted isomorphisms and observational equivalence.
- **Intensional models**, characterize the $\beta\eta$ theory and provable isomorphisms.

Proof equivalence in *MLL*2

As for System F, separability fails: these two proof nets are non separable:



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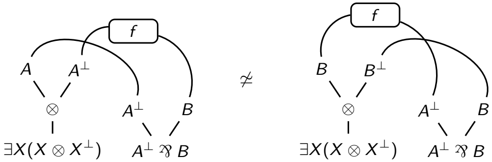

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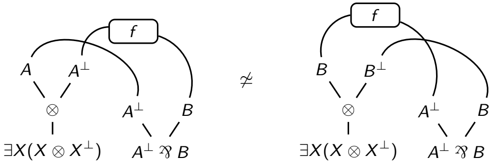

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However, Seiller and Nguyen recently proved that observational equivalence in *MLL*2 is decidable and has finitely many classes at each type.

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- the "wanted" isomorphism $\top \simeq \exists XX$ says that all proofs of $\exists XX$ are equal: hence *MLL*2 has some "additive" behavior (initial and terminal objects)

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In general, all such new equivalences do not preserve witnesses of $\exists$. Hence we need new approaches to the syntax and semantics of the $\exists$-rule!

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$MLL2$ and the coend calculus

$MLL2$ formulas correspond to multivariant functors $F, G : \mathbb{C}^{op} \otimes \mathbb{C} \to \mathbb{D}$, e.g. $\mathbb{C}(X, X)$, $X \otimes X^{\perp}$.

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Limits and colimits of dinaturals are given by **ends** and **coends**

$$\begin{tikzcd}
C \arrow[dr, dashed, "h"] \arrow[rr, bend left, "\theta_A"] \arrow[ddr, bend right, "\theta_B"'] & & \\
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\qquad
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F(B,A) \arrow[r, "F(f,A)"] \arrow[d, "F(B,f)"] & F(A,A) \arrow[d, "\omega_A"] \arrow[ddr, bend left, "\eta_A"] & \\
F(B,B) \arrow[r, "\omega_B"'] \arrow[drr, bend right, "\eta_B"] & \int^X F(X,X) \arrow[dr, dashed, "h"] & \\
& & C
\end{tikzcd}$$

Essentially, quantifiers + **equalizer/co-equalizer conditions**

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From this we can deduce all “wanted” isos:

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- connectives: $\int_X ((A \multimap B \multimap X) \multimap X) \simeq A \otimes B$
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If $\mathbb{C}$ is *-autonomous and complete, we can extend the interpretation to *MLL*2 as follows:

- $(\forall X A[X])^{\mathbb{C}} := \int_X A^{\mathbb{C}}(X, X)$, $(\exists X A[X])^{\mathbb{C}} := \int^X A^{\mathbb{C}}(X, X)$
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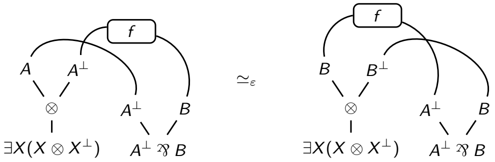
We let then

$$\pi \simeq_\varepsilon \sigma \text{ if for all } \mathbb{C} \text{ } ^*\text{-autonomous and complete } \pi^{\mathbb{C}} = \sigma^{\mathbb{C}}$$

$\simeq_\varepsilon$ is then a congruence which extends $\beta\eta$, due to equalizer/co-equalizer conditions.

Coends and non separable proof nets

Equalizer/co-equalizer conditions capture non-witness-preserving permutations:



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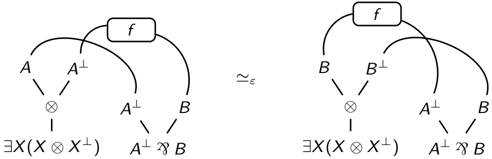

is just a coend diagram:

$$\begin{tikzcd} B^\perp \otimes A \arrow[r, "f^\perp \otimes A"] \arrow[d, "B^\perp \otimes f"] & A \otimes A^\perp \arrow[d, "\omega_A"] \\ B \otimes B^\perp \arrow[r, "\omega_B"'] & \int^X X \otimes X^\perp \end{tikzcd}$$

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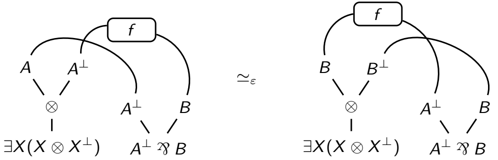

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*MLL*2 proof nets are not dinatural

Similarly, the following equation in linear natural deduction

$$
\dfrac{\dfrac{\dfrac{\dfrac{\forall X(X \multimap X)}{A \multimap A} \qquad [\overset{n}{A}]}{A}}{\overset{f}{B}}}{A \multimap B}\, n
\quad \simeq_{\varepsilon} \quad
\dfrac{\dfrac{\dfrac{\forall X(X \multimap X)}{B \multimap B} \qquad \begin{matrix}[\overset{n}{A}]\\ f \\ B\end{matrix}}{B}}{A \multimap B}\, n
$$

is just an end diagram:

$$
\begin{tikzcd}
\int_X \mathbb{C}(X,X) \arrow[r, "\delta_A"] \arrow[d, "\delta_B"] & \mathbb{C}(A,A) \arrow[d, "\mathbb{C}(A,f)"] \\
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\end{tikzcd}
$$

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Yoneda formulas

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Yoneda isomorphisms in *MLL*2:

$$\int_X ((\bigotimes_i^n C_i \multimap X) \multimap D[X]) \simeq D[\bigotimes_i^n C_i \otimes \mathbf{1}_{\mathcal{Y}}]$$

$$\int^X ((\bigotimes_i^n C_i \multimap X) \otimes D^\perp[X^\perp]) \simeq D^\perp[\bigparr_i^n C_i^\perp \parr \perp_{\mathcal{Y}}]$$

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$MLL2_{\mathscr{Y}}$ is the fragment of $MLL2$ in which quantification is restricted to Yoneda formulas:

- $\forall XA$ is admitted only if A is **Yoneda in X**: $A = (\bigotimes_i^n C_i \multimap X) \multimap D[X]$
- $\exists XA$ is admitted only if A is **co-Yoneda in X**: $A = (\bigotimes_i^n C_i \multimap X) \otimes D[X]^{\perp}$

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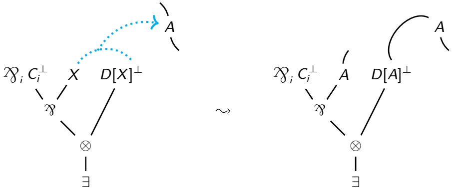

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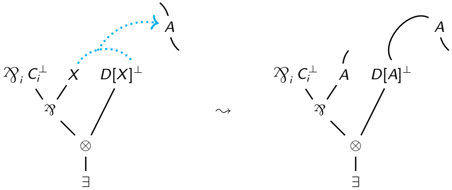

Correct $\exists$-linkings are considered up to **rewitnessing**: $\simeq_w$ is the refl./trans. closure of 1-rewitnessing (i.e. changing W by W' differing only by one value).

From Trimble rewiring to rewitnessing

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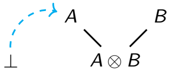

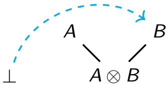

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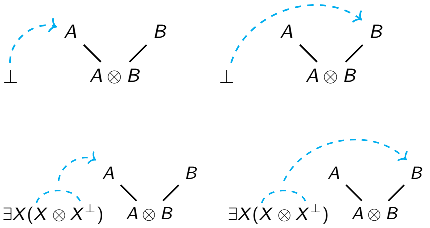

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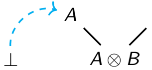

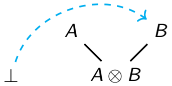

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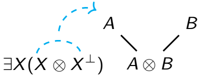

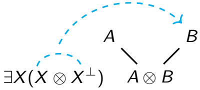

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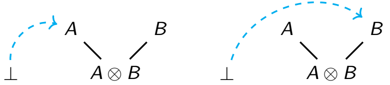

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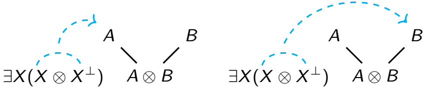

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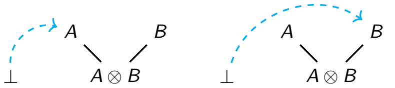

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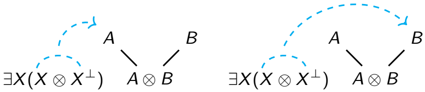

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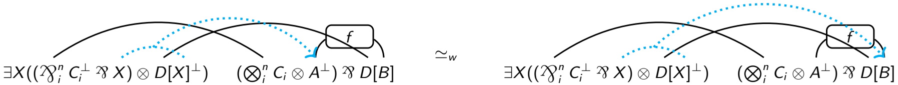

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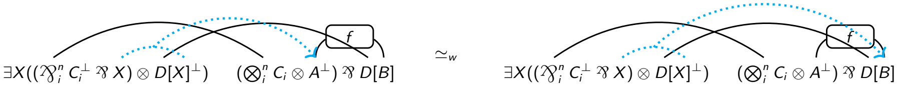

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Theorem 3: for $MLL2_{\mathscr{Y}}$, the category of $\exists$-linkings modulo rewitnessing is isomorphic to the category of proof nets modulo $\simeq_\varepsilon$.

Outline

1. Proof nets and proof equivalence
2. Proof nets, coends and the Yoneda isomorphism
3. Weak coherence for coends
4. Observational equivalence (joint work with L. Tortora de Falco)

Observational equivalence

We consider the equivalence $\simeq_{Obs}$ defined as follows:

$\pi : A \ \simeq_{Obs} \ \sigma : A$ when for all P propositional and $\delta : A^{\perp}, P$, $[\pi, \delta] \simeq_{\beta\eta} [\sigma, \delta]$

In other words, we use *MLL* proof nets as **observables**

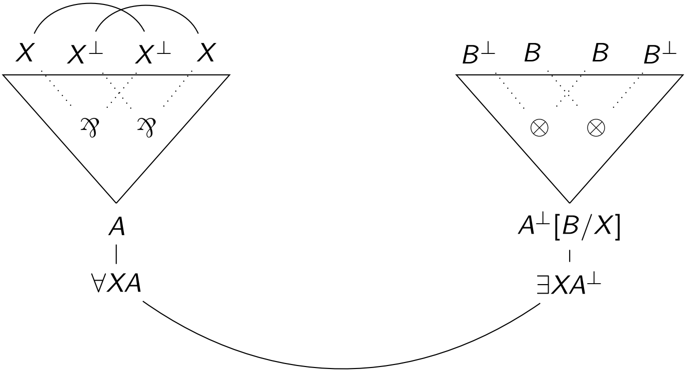
The proof nets $\delta : A^{\perp}, P$ are the **observations**

Remarks:

- many proof nets $\pi : A$ have no observations (e.g. $A = \exists X X, \exists X(X \multimap X)$)
- hence for such formulas $\simeq_{Obs}$ is trivial.
- $\simeq_{Obs}$ includes $\simeq_{\varepsilon}$ strictly

Characterising equivalence through *MLL* proof nets

Cut-elimination in *MLL*2: transporting a $\parr$-**linking** onto a $\otimes$-**linking**:



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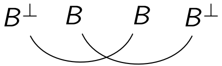

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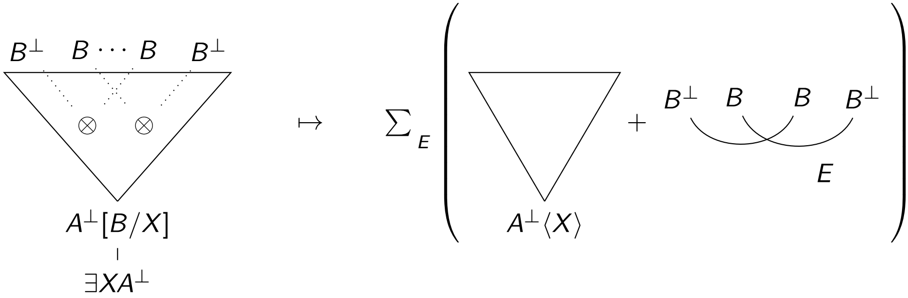
Characterising equivalence through *MLL* proof nets

This allows to define a translation $\pi \mapsto \sum_i \pi_i$ from *MLL*2 to formal sums of *MLL* proof nets:



Where E varies among the $\parr$-linkings of A.

Characterising equivalence through *MLL* proof nets

This procedure allows to eliminate all $\exists$-links and yields a finite set of *MLL* proof nets.

$$\pi \;\mapsto\; \sum_i \pi_i$$

Remarks:

- when a $\parr$-linking does not exists (e.g. $\forall XX$), the construction yields the empty set (e.g. all proofs of $\exists XX$ are equally empty).
- however, when $\pi^\circ : A$ is empty, then it means it has no observables: no $\sigma : A^\perp, P$, where P is propositional.

Characterising equivalence through *MLL* proof nets

The interaction between a proof and an observation is characterized by the *MLL* translation:

Lemma. If $\pi : A$ and $\delta : A^{\perp}, P$, then there exist unique i, j in the *MLL* translations of π and σ such that

$$[\pi, \delta] \ \simeq_{\beta} \ [\pi_i, \delta_j]$$

From this, and the usual characterization of $\simeq_{Obs}$ for *MLL*, we get

Theorem. The *MLL* translation captures observational equivalence, i.e.

$$\pi \simeq_{Obs} \sigma \quad \textit{iff} \quad \pi^{\circ} = \sigma^{\circ}$$

A "finite" relational model for *MLL*2

The *MLL* translation induces an extension of the usual relational model of *MLL* to *MLL*2 satisfying

$$[\![\pi]\!] = \bigcup_i [\![\pi_i]\!]$$

- formulas correspond to certain polynomial functors $\Phi(\vec{x}) = \Pi_{i,j} x_i^{n_i} \cdot c_i^{m_i}$ (where the constants c_i stand for bound variables)
- proofs correspond to multi-graphical relations, i.e. family of relations $\theta_{\vec{x}}$ essentially induced by a finite set of *MLL* proof nets: for all sets $\vec{a}$

  $$p_i(\theta_{\vec{a}}) = p_j(\theta_{\vec{a}}) \quad when \quad (i,j) \in \mathscr{G}$$

  and $\mathscr{G}$ is some allowable graph (equivalently, a *MLL* proof net).
- this gives rise to a *-autonomous fibration $\mathbf{MG} \to \mathbf{P}$, with adjoints

  $$\Sigma \dashv \pi^* \dashv \Pi$$

  precisely corresponding to interpretation of $\forall, \exists$ as finite sets of $⅋$-linkings/$\otimes$-linkings.

Conclusions

We introduced two approaches to capture proof equivalence in *MLL*2 by a different interpretation of the $\exists$-link:

- by interpreting $\exists$ as a **coend**: we characterized the equivalence $\simeq_\varepsilon$ induced by coends by rewitnessing, a variant of Trimble's rewiring for a fragment of *MLL*2 related to the Yoneda isomorphism.

- by analyzing the behavior of $\exists$ through cut-elimination, we defined a translation $\pi \mapsto \sum_i \pi_i$ from *MLL*2 proof nets to finite sets of *MLL* proof nets which
  - characterizes observational equivalence
  - leads to a "finite" relational model for *MLL*2 characterizing observational equivalence too.

Future work:

- Rewitnessing beyond Yoneda formulas (e.g. how to treat initial algebras?)
- Computing coends isomorphisms through proof nets?
- Observational equivalence for *MELL*2? Interaction between the *MLL* translation and Taylor expansion?

Thank you !